Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

GROUPS WHICH CONTAIN AN ABELIAN SUBGROUP OF PRIME INDEX.

By G. A. Miller.

§ 1. Introduction.

A necessary and sufficient condition that an abelian group contains a subgroup of index p, p being a prime number, is that the order of this group is divisible by p. In what follows we shall therefore assume that the group G under consideration is non-abelian, and we shall use the symbol H to represent an abelian subgroup of index p contained in G. From the fact that every subgroup of prime index in any group is maximal, it results that H is a maximal subgroup of G. If H is non-invariant any two conjugates of H must therefore generate G, and the common operators of these two conjugates must constitute the central of G. In fact, it is evident that the common operators of any two conjugate abelian maximal subgroups* must always constitute the central of a group, but the central is not necessarily contained in two such subgroups.

Every subgroup of index ρ under any group includes $1/\rho'$, $\rho' \overset{=}{<} \rho$, of the operators of every other subgroup of this group. We proceed to prove that $\rho' < \rho$ whenever the two subgroups in question are conjugate and to deduce a few results from this theorem. Suppose that K_1, K_2 are two conjugate subgroups of index n under the group K, and let K_0 represent the subgroup formed by all the common operators of K_1 and K_2. Since the index n' of K_0 under K_1 cannot exceed n, it is only necessary to prove that we arrive at an absurdity by assuming $n' = n$. If n' were equal to n all the operators of K could be represented in the form $s_1 s_2$, where s_1 represents any operator of K_1, and s_2 represents any operator of K_2. It results therefore that all the conjugates of K_1 would be transforms of K_1 under K_2. This is impossible since the operators of K_2 cannot transform K_1 into K_2. It has therefore been proved that $n' < n$. That is, *the index of the subgroup formed by the common operators of two conjugate subgroups, under either of these subgroups, is always smaller than the index of one of these two conjugate subgroups under the entire group.*

From the theorem which has just been quoted it is very easy to derive the known theorem that every subgroup of index p in a group of order p^m

---

* It should be observed that an abelian maximal subgroup is also a maximal abelian subgroup but the converse is not necessarily true.

is invariant. If such a subgroup were not invariant there would be two conjugate subgroups of this order. The common operators of these subgroups would constitute a subgroup whose index would be less than p under each of these conjugate subgroups. As this index must also be a divisor of p^{m-1} it must be unity. That is, the two subgroups in question must be identical, and hence every subgroup of index p in a group of order p^m is invariant. This theorem is also known to be a special case of the following theorem. In a group of order p^m every non-invariant subgroup is transformed into itself by at least p of its conjugates.*

The theorem in italics includes also the well known theorem that a subgroup of index 2 is necessarily invariant. If there were two conjugate subgroups of this index their common operators would constitute a subgroup whose index under each of these two conjugate subgroups would be less than 2, and hence this index would be unity. It is also evident that two conjugate subgroups of index 3 must always have exactly half their operators in common, and these common operators constitute an invariant subgroup of the entire group.

It results directly from the properties of a direct product that the index of the subgroup formed by the common operators of two invariant subgroups, under either of these subgroups, is *not* always smaller than the index of one of these subgroups under the entire group. In fact, if a group G contains two invariant subgroups G_1, G_2 of the same prime index p, all the common operators of G_1, G_2 constitute an invariant subgroup G_0 of G, since all the common operators of any two invariant subgroups constitute an invariant subgroup. As p divides the orders of all the operators of G which are not contained in G_0, we have proved the theorem: *The common operators of two invariant subgroups of the same prime index p constitute an invariant subgroup of index p under each of these two invariant subgroups.*

§ 2. Groups in which an abelian subgroup of prime index is non-invariant.

Since H is a non-invariant subgroup of index p under G the central of G must be a subgroup of H whose index under H is less than p and $p > 2$. Hence the p conjugates of H are transformed under G according to the metacyclic group of order $p(p-1)$ or according to an invariant subgroup of this group. This transformation group is evidently simply isomorphic with the central quotient group of G, and hence it follows that the index of the central of G under H is a divisor of $p-1$. If i represents this index, the order of the cental quotient group of G is ip, and it is evident that any possible isomorphism between the group of order ip contained in the metacyclic group of order $p(p - 1)$ and any abelian group whatever will result in a group which involves an abelian non-invariant subgroup of index p. It should be observed that direct products are to be included among these possible isomorphisms. We proceed to prove that, conversely, every possible group which involves non-invariant abelian subgroups of index p can be obtained by establishing such an isomorphism.

---

* American Journal of Mathematics, vol. 23 (1901), p. 173.

To prove this theorem we may first observe that each of the Sylow subgroups of G is abelian and that the independent generators of the Sylow subgroup of order p^m may be so selected that all except one of them are in the central of G. The one which is not in this central is of order p and generates the commutator subgroup of G. In fact, if s is any operator of G which is not in the abelian group generated by the central of G and the Sylow subgroup of order p^m, it follows that s transforms a cyclic subgroup of the Sylow subgroup of order p^m into itself, since the number of the cyclic subgroups of order p^α, in any group of order p^m, which do not have their generators in a given non-cyclic subgroup of order p^{m-1} is always a multiple of p, $p > 2$.* Since the order of s is prime to p and since the pth powers of the generators of the cyclic subgroup of order p^α which s transforms into itself are invariant, it results that $\alpha = 1$ and that this cyclic subgroup is the commutator subgroup of G, as was observed above.

From the preceding paragraph it follows that whenever the order of G is divisible by p^m, $m > 1$, G must be a direct product of an abelian group of order p^{m-1} and of some other factor group. It also results that all the Sylow subgroups of G whose orders are prime to ip, if such Sylow subgroups exist in G, are *proper factors* of G; that is, G is the direct product of these factor groups and of some other subgroup. Hence it results that *G may be constructed by establishing an isomorphism between an abelian group and the group of order ip contained in the metacyclic group of order $p(p - 1)$, whenever G contains a non-invariant abelian subgroup of index p.*

It has been observed that G is divisible whenever any one of its Sylow subgroups is contained entirely in its central, and also when the order of G is divisible by p^2. Suppose that G does not satisfy either of these two conditions. Hence G may be obtained by establishing an isomorphism between an abelian group G_1, whose order does not involve any factor which is prime to i, and the invariant subgroup G_2 of order ip contained in the metacyclic group of order $p(p - 1)$. It is also clear that G_1 is cyclic whenever G is indivisible since the Sylow subgroups of G_2 are cyclic. Moreover, G cannot be indivisible when G involves the direct product of any Sylow subgroup of G_1 and a Sylow subgroup of G_2, or when the order of a Sylow subgroup of G_1 is equal to or less than the order of the corresponding

---

* Proceedings of the London Mathematical Society, series 2, vol. 2 (1905), p. 142.

Sylow subgroup of G_2 and the isomorphism between these Sylow subgroups is α, β, where $\alpha > 1$. Hence we are led to the following theorem: *In order that an indivisible group G involves a non-invariant subgroup of index p, it is necessary and sufficient that the following three conditions are satisfied: G may be constructed by establishing an isomorphism, between a group G_2 of order ip, $i > 1$, contained in the metacyclic group of order $p(p-1)$ and a cyclic group G_1 whose order involves no prime factors except those of i; in this isomorphism there is a $(1, \alpha)$ correspondence between the operators of two Sylow subgroups contained in G_1 and G_2 respectively, whenever the order of the Sylow subgroup of G_1 does not exceed the order of the corresponding Sylow subgroup of G_2; the isomorphism between two Sylow subgroups of G_1 and G_2 respectively is not a direct product unless the former of these subgroups is the identity.*

§ 3. **Groups in which an abelian subgroup of prime index is invariant.**

It results from the preceding section that G cannot involve both an invariant and also a non-invariant abelian subgroup of the same prime index. From the theorem proved at the end of the Introduction it follows that the non-abelian group G contains exactly $p + 1$ abelian subgroups of prime index p whenever it contains more than one such subgroup. When G contains $p + 1$ such subgroups the central quotient group of G is the non-cyclic group of order p^2, and the common operators of these $p + 1$ invariant abelian subgroups constitute the central of G. As H is the direct product of its Sylow subgroups, *G is the direct product of an abelian group whose order is prime to p and of a non-abelian group of order p^m, $m > 2$, whenever G contains more than one invariant abelian subgroup of index p. This non-abelian group of order p^m contains exactly $p + 1$ abelian subgroups of order p^{m-1}.*

As every direct product of an abelian group and of a non-abelian group of order p^m which contains more than one abelian subgroup of order p^{m-1}, evidently constitutes a non-abelian group involving more than one abelian subgroup of index p, it results that a necessary and sufficient condition that a non-abelian group contains more than one abelian invariant subgroup of index p is that it satisfies the conditions imposed on G in the theorem of the preceding paragraph. It also results from this theorem and from those of the preceding section that every non-abelian group which contains more than one abelian subgroup of index p has a commutator subgroup of order p, and these abelian subgroups are invariant or non-invariant as the commutators of order p are invariant or non-invariant under the entire group. As a very elementary illustration of this theorem it may be observed that the dihedral group of order $2p$ contains non-invariant commutators of order p and hence its abelian subgroups of index p are non-invariant.

In what follows we shall assume that H is cyclic and we shall first consider the case when $p = 2$. It will be convenient to assume that the order h of H is written in the form

$$h = 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\lambda^{\alpha_\lambda},$$

where $p_1, p_2, \cdots, p_\lambda$ are distinct odd prime numbers and all the exponents $\alpha_1, \alpha_2, \cdots, \alpha_\lambda$ exceed 0. Since the group of isomorphisms of the cyclic group of order p^m, $p > 2$, involves one and only one operator of order 2 it results that the number of possible non-abelian G's, when $\alpha_0 = 0$, is given by the formula

$$\lambda + \frac{\lambda(\lambda - 1)}{2!} + \frac{\lambda(\lambda - 1)(\lambda - 2)}{3!} + \cdots + \lambda + 1 = (1 + 1)^\lambda - 1 = 2^\lambda - 1.$$

It may be observed that the number of these groups depends only on λ and not upon the values of $\alpha_1, \alpha_2, \cdots, \alpha_\lambda$. The formula follows directly from the fact that the operator of order 2 which must be adjoined to H to generate G may transform the Sylow subgroups of H in as many different ways as the integer expressed by this formula.

When $\alpha_0 = 1$, H involves an operator of order 2 which is invariant under G and may be either dihedral or dicyclic. There are therefore just twice as many possible groups in this case as in the case which has just been considered. That is, the number of possible non-abelian groups when $\alpha_0 = 1$ is given by the formula $2^{\lambda+1} - 2$. When $\alpha_0 = 2$ the group of isomorphism of the Sylow subgroup of order 4 in H involves also one operator of order 2, and hence the number of possible non-abelian groups in this case is $2^{\lambda+2} - 2$.

It remains to consider the case when $\alpha_0 \overline{\gt} 3$. In this case we have to add to the groups just considered those which result when the operators of the Sylow subgroup of order 2^{α_0} in H are transformed either into their $2^{\alpha_0-1} - 1$ or into their $2^{\alpha_0-1} + 1$ powers. That is, we have to add twice the number of the possible groups when $\alpha_0 = 0$, including the case when the group is abelian. Hence the possible number of non-abelian groups in this case is given by the formula

$$2^{\lambda+2} - 2 + 2^{\lambda+1}.$$

As the number of the possible abelian groups which contain a cyclic subgroup of order h is 2 when h is even and only one when h is odd we have established the following theorem, which includes abelian groups: *If* λ *represents the number of the distinct odd prime factors of the order* h *of a cyclic group* H, *there are exactly* 2^λ *groups of order* $2h$ *which include* H *whenever* h *is an odd number, and there are exactly* $2^{\lambda+1}$ *such groups when* h *is the double of an odd number. When* h *is divisible by* 4 *but not by* 8 *the number of these* *possible groups is* $2^{\lambda+2}$, *and this number is* $3.2^{\lambda+1}$ *when* h *is divisible by* 8. Although this theorem was developed under the assumption that $\lambda > 0$ it is clearly true also when $\lambda = 0$. It may be observed that the number of these distinct groups increases rapidly with the number of the distinct prime factors of the order of the given cyclic group. For instance, there are 24 distinct groups of order 240 which include a common cyclic subgroup of order 120.

When the prime index p of the invariant cyclic subgroup H contained in the non-abelian group G is odd, it follows that either p^2 divides h or that p divides at least one of the numbers $p_1 - 1$, $p_2 - 1$, $\cdots$, $p_\lambda - 1$. If p^2 divides h, we let $\gamma - 1$ represent the number of the numbers $p_1 - 1$, $p_2 - 1$, $\cdots$, $p_\lambda - 1$ that are divisible by p, and, if p^2 does not divide h, we let γ represent this number. As the group of isomorphisms of the cyclic group of order p^m is cyclic and of order $p^{m-1}(p - 1)$, while the group of isomorphisms of the cyclic group of order 2^m is of order 2^{m-1}, it is easy to see that the number of the distinct possible non-abelian groups of order hp which include H is given by the following formula:

$$\gamma + \frac{1}{2!}(p-1)\gamma(\gamma-1) + \frac{1}{3!}(p-1)^2\gamma(\gamma-1)(\gamma-2) + \cdots + (p-1)^{\gamma-1}.$$

If we combine this formula with the theorem of the preceding paragraph and observe that the number of abelian groups of order hp, which include H, is 2 or 1 according as h is or is not divisible by p, we obtain the total number of the groups of order hp which include the cyclic group H as an invariant subgroup of prime index.